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"Escape" of a Periodically Driven Particle from a Metastable State in a Noisy System

Moshe Gitterman[1,2] **and George H. Weiss**[1]

We consider the statistical properties of the escape time of a particle initially sitting in a potential well subject to a combination of white noise and a periodic forcing term. As one finds in the case of the much-studied bistable potential, different kinds of resonant effects can occur, as measured by the survival probability and the average residence time. When this time is considered as a function of the noise strength, then we show that for small amplitudes of the forcing term there are no resonant effects, while for large amplitudes such effects can appear. We also show that a resonant phenomenon is possible in terms of the amplitude of a periodic forcing term.

KEY WORDS: Dynamical systems; metastable states; first passage times.

1. INTRODUCTORY REMARKS

Stochastic resonance, i.e., a variety of resonant effects attributable to the interaction between a periodic applied force and noise in nonlinear dynamical systems, has been identified in a number of physically realizable systems.[1-4] One can distinguish between two types of resonance in classical mechanics, the first referring to the type occurring in linear systems and the second to that in nonlinear systems. If a linear system is subjected to a sinusoidal forcing term of the form $A \cos \omega t$, where A is an amplitude, resonance occurs when ω is equal to a natural frequency of the physical system, independent of the value of the amplitude. On the other hand, when the dynamical system is nonlinear the resonance phenomenon can indeed be a function of the amplitude as well as the frequency. Analogous phenomena exist when the system is also influenced by noise.

[1] Physical Sciences Laboratory, Division of Computer Research and Technology, National Institutes of Health, Bethesda, Maryland 20892.

[2] Permanent address: Department of Physics, Bar-Ilan University, Ramat-Gan, Israel 52100.

Among other results in the present paper we will show that both the frequency and the amplitude can play significant roles.

By now there is a large literature devoted to elucidating properties to be expected in systems with interacting effects of noise and a periodic forcing term. A majority of the papers on this subject emphasize resonant properties associated with the signal-to-noise ratio and the correlation function, but more recently some attention has been paid to the study of resonance phenomena in the kinetic properties of primarily nonlinear dynamical systems. As an example we mention the existence of a series of peaks in the probability density of the escape time in a bistable system.[5] Different kinds of resonance phenomena produced by a periodic driving field have been shown to exist in diffusive systems in the presence of both trapping[6] and reflecting[7] boundaries.

In this article we consider still another manifestation of the interaction between periodic and random forces for a nonlinear system. Specifically, this system consists of a single potential well, and the resonant properties appear in the behavior of a function that we define as an average escape time. Let us consider a particle moving in one dimension subject to a potential field which is chosen to have the form

$$U(x, t) = -\frac{x^2}{2} + \frac{x^3}{3} + \frac{A}{4} x \cos(\omega t) \tag{1}$$

which is presumed to act on particle in addition to zero-mean white noise $n(t)$, whose first two moments satisfy $\langle n(t) \rangle = 0$ and $\langle n(t)\, n(t') \rangle = 2D\delta(t-t')$. In the preceding equations A and D are constants. By convention A is chosen to be positive. The potential in Eq. (1) has no more than a single minimum, and a permanent escape from the potential well is possible both with and without additive noise. Such a system differs from the bistable potential, which is often characterized by a steady state. In consequence the set of parameters likely to be of interest can differ from those used in the study of the bistable system.

If we neglect inertial effects (that is, the system is overdamped), the equation of motion of a particle moving under the joint influence of noise and the periodic driving force in Eq. (1) is

$$\frac{dx}{dt} = x - x^2 + \frac{A}{4} \cos(\omega t) + \sqrt{D}\, n(t) \tag{2}$$

The Fokker–Planck equation satisfied by the probability density, $p(x, t \mid x_0)$, for the position of a particle at time t that was initially at x_0 is

$$\frac{\partial p}{\partial t} = D \frac{\partial^2 p}{\partial x^2} - \frac{\partial}{\partial x} \left\{ \left[x - x^2 + \frac{A}{4} \cos(\omega t) \right] p \right\} \tag{3}$$

When the potential includes neither noise nor a periodic forcing term, that is, $A = D = 0$ in Eqs. (2) and (3), it will have a maximum at $x = 0$ and a minimum at $x = 1$, thereby defining a potential well of the form shown in Fig. 1.

This well persists at all values of the time provided that $A \leqslant 1$, although the positions of the minimum and maximum will shift from the points $x = 0$ and 1. When $A \leqslant 1$ it is easily shown that the trajectory of a particle initially within the well will always remain bounded. When these conditions are not satisfied the particle can escape over the barrier and for certain combinations of parameters will satisfy the condition $\lim_{t \to \infty} x(t) = -\infty$. The satisfaction of this limiting behavior will define what we mean by the term escape. It is quite difficult to calculate the parameter combinations that ensure escape for motion described by Eq. (2) since it involves a knowledge of quantitative properties of the Mathieu equation.

The addition of noise to the picture enables a particle to escape from the well no matter what the value of the parameter A in Eqs. (2) and (3). We have not been able to determine whether escape occurs with a probability equal to one, but conjecture that this is indeed the case.

If $D \neq 0$ but $A = 0$, which is to say that the system is subject to noise but not to a periodic forcing term, the rate of escape of a particle from a potential well predicted by the Kramers theory in the case of the high-friction limit is[8]

$$k_0 = \frac{\sqrt{2}}{\pi} [|U''(x=0)| U''(x=1)]^{1/2} \exp\left(-\frac{\Delta U}{D}\right) = \frac{\sqrt{2}}{\pi} \exp\left(-\frac{1}{6D}\right) \tag{4}$$

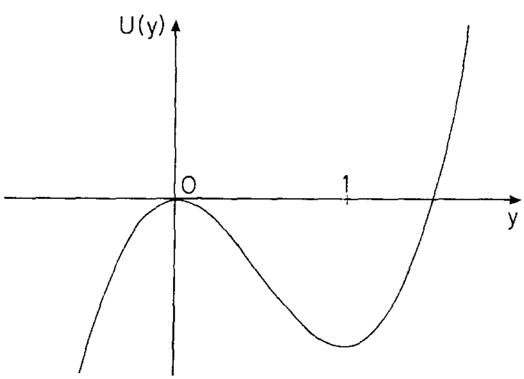

Fig. 1. A schematic diagram of the part of the potential that is independent of time.

where ΔU $(=1/6)$ is the barrier height in Fig. 1. The Kramers derivation of Eq. (4) presumes that the system is overdamped, which implies that the magnitude of the noise is required to satisfy

$$D \ll 1/6 \tag{5}$$

which will be assumed in much of the analysis to follow. Because we have not been able to settle the question of the conditions under which complete escape occurs, we will only be able to discuss a weaker version of this general phenomenon. This version defines escape as being equivalent to the particle reaching the peak of the barrier, which, of course, is not equivalent to the statement that $x(\infty) = -\infty$. However, it is obviously a necessary condition for this to occur. Within this simplified version of the escape problem it will prove possible to discuss parameters exemplified by the survival probability and mean escape time as a function of the amplitudes of the forcing term, the noise, and the driving frequency. By survival probability we will mean the probability that the particle remains within the well and has never reached the maximum. The Kramers analysis implies that at sufficiently long times the survival probability $S(t)$ can be approximated by a negative exponential which we write

$$S(t) \sim e^{-k_0 t} \tag{6}$$

Notice that we have not written a full equality sign here because at early times the particle will remain in the potential well and the detailed structure of the potential barrier is then important. The exponential form in (6) arises because the particle spends a considerable amount of time in the neighborhood of the maximum. Several authors replace the factor k_0 by $k_0/2$ to take into account that the particle, when at the top of the barrier, may move in both directions, either escaping to $-\infty$ or returning to the well. This distinction is unimportant in the context of our investigation.

We will be interested in the qualitative changes induced in both the rate and the survival probability when the system is subject to the combination of a periodic modulating field and white noise. Without any detailed calculations we can expect to see at least some ripples in $S(t)$ due to the periodic force [but remember that $S(t)$ must be a nonincreasing function]. Indeed, the deterministic forcing term in Eqs. (2) and (3) periodically induces the particle to approach the barrier peak when $\cos(\omega t)$ is negative. It remains to be seen whether this change in the behavior of $S(t)$ leads to nonmonotonic behavior of the mean survival time $\langle t \rangle$ considered as a function of D. Our results indicate that this does not occur for small A, but does occur when A is large. The precise definition of what we understand by large and small A will be given shortly. As a parenthetic remark, our calculations indicate that $\langle t \rangle$ considered as a function of A can be nonmonotonic even for small A. A complete solution to this general question involves some rather tricky mathematical problems, but one can make partial progress in the study of various limiting cases which shed some light on the behavior expected in this class of systems. In the following section we briefly review some results presented in ref. 9 on the effects of a periodic field on the dynamics of escape from the potential well defined by Eq. (1) with $D = 0$. This case is sufficiently simple so that one can obtain precise results.

2. THE NOISE-FREE CASE

When there is neither a periodic forcing term nor noise the solution to Eq. (2) is

$$x(t) = \frac{x(0)}{x(0) + [1 - x(0)]e^{-t}} \tag{7}$$

from which it follows that escape occurs if and only if $x(0)$ is negative and it occurs instantaneously at the time

$$t = \ln\left(1 + \frac{1}{|x(0)|}\right) \tag{8}$$

Let us next add a sinusoidal forcing term $(A \neq 0)$ in Eq. (2) while keeping $D = 0$ and consider first the case in which escape does not occur, i.e., when the trajectory has both a bounded minimum and a bounded maximum. In order to have such a trajectory we require that $\dot{x} = 0$ at two real values of $x(t)$ which can be calculated by setting the right-hand side of Eq. (2) equal to zero and requiring that the roots of the resulting equation be real. This consideration implies that the trajectory is bounded, or equivalently that the particle does not escape from the well provided that $x(0)$ satisfies the inequality

$$x(0) \geqslant \tfrac{1}{2}[1 - (1 + A)^{1/2}] \tag{9}$$

When $D = 0$ and $A \neq 0$ we can, in fact, discuss escape in the rigorous sense $[x(\infty) = -\infty]$, but must define what we mean by the escape time, since there cannot be a specific value at which the particle escapes as there is in Eq. (8).

One can find a sufficient condition for escape from the well by dividing the time into successive cycles defined by the period of the sinusoidal forcing term. In the nth cycle the particle trajectory has both a maximum and a minimum value, the minimum occurring when $\cos(\omega t) = -1$, i.e., at times $t = (2n-1)\pi/\omega$, where $n = 1, 2,\ldots$. In an arbitrary cycle call the value of $x(t)$ at such a minimum x_m. Escape occurs provided that both x_m and the velocity at x_m are negative. This is equivalent to the requirement that

$$-2x_m > (1+A)^{1/2} - 1 \tag{10}$$

Figure 2 shows curves of $x(t)$ for a number of frequencies of the sinusoidal term, illustrating the fact that escape, as we have defined it, can occur at the end of any cycle, and the escape cycle is very sensitive to the frequency.

The condition in (10) also indicates that escape is not an automatic consequence of the condition $A > 1$. On defining $T = 2\pi/\omega$ as the cycle time, one can show that the particle cannot escape when $x(T) > x(0)$. To see this, set $\varepsilon(t) = x(t+T) - x(t)$. This function satisfies

$$\dot{\varepsilon} = (1-2x)\varepsilon - \varepsilon^2 \tag{11}$$

The argument is based on the assertion that the condition $\varepsilon(0) > 0$ implies that $\varepsilon(nT) > 0$ for all integer n. To see this, assume that $x(t) \neq 1/2$ (this proves to be no real restriction) and suppose that $\varepsilon(t)$ is small enough so that the quadratic term in Eq. (11) can be neglected. The solution to the resulting linear equation is

$$\varepsilon(t) = \varepsilon(0) \exp\left[t - \int_0^t x(\tau)\, d\tau\right] \tag{12}$$

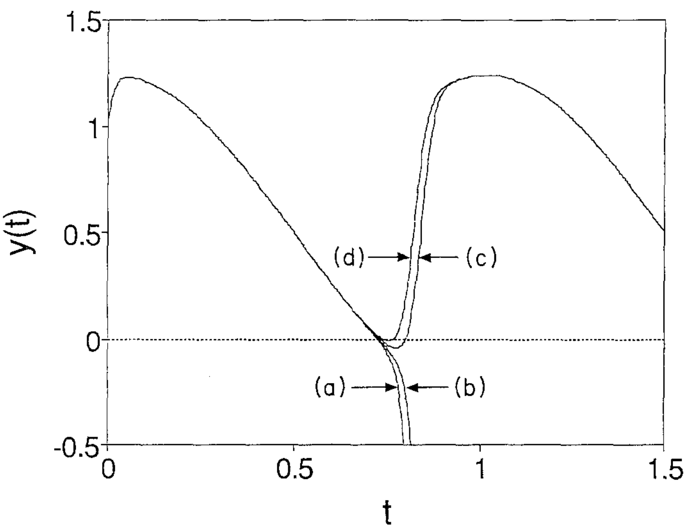

Fig. 2. Trajectories of a particle whose dynamics is described by Eq. (2) plotted as a function of time. The parameters used are $D = 0$ and $A = 1.2$ and the frequencies are (a) $\omega = 0.13196$, (b) $\omega = 0.13197$, (c) $\omega = 0.13198$, (d) $\omega = 0.13199$.

which implies that $\varepsilon(T)$ must have the same sign as $\varepsilon(0)$. By induction one finds that when $\varepsilon(0)$ is positive $\varepsilon(nT)$ is likewise positive, which implies that the particle cannot escape from the well without being subject to the additive noise term indicated in Eq. (2).

When $x(T) < x(0)$ we cannot be sure that the particle will ultimately escape from the well, hence $\varepsilon(T) > 0$ is a sufficient, but not a necessary condition for the particle to remain trapped. These considerations do not tell us what to expect when $\varepsilon(0)$ is negative except when the particle is at the top of the barrier, when one can show that escape is certain provided that $x(T)$ is out of the well. Some qualitative aspects of the escape time for different values of A and T are shown in Fig. 3 for this particular case. When A and T are points lying above the upper curve escape occurs even before the end of the first cycle. The area between the upper curve and the next closest one contains (A, T) pairs for systems in which the particle escapes at the end of the second cycle, and so on. The points lying below the lowest of the curves correspond to systems in which there is no escape.

In the following sections we turn to the main part of our analysis concerning the relation between the noise and the periodic term, which will be seen to be different, depending on the magnitudes of both A and T.

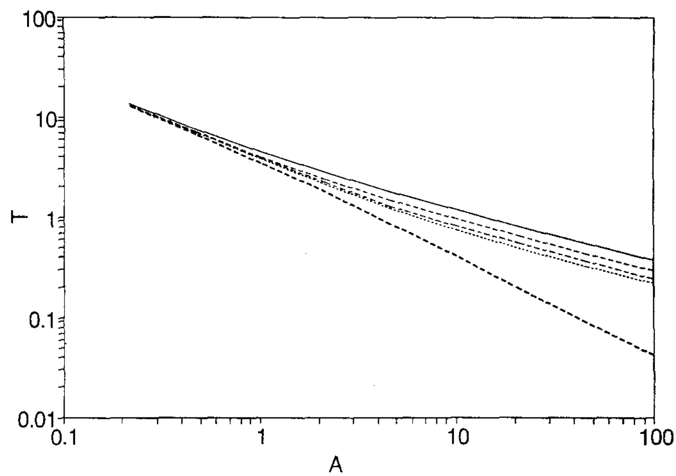

Fig. 3. Separatrices in the (A, T) plane that indicate the cycle number at which escape occurs. The region above the upper curve corresponds to escape before the first cycle, the region between the upper and adjacent curves to escape between the first and second cycles, and so forth. The region below the lowest curve corresponds to trapped particles.

3. THE ADIABATIC APPROXIMATION; $\omega \ll 1$

The adiabatic approximation is defined by the condition that a particle inside the well comes to equilibrium quickly in comparison with a period of the external field. All of the following analysis will be based on the assumption that a potential well exists at all values of the time, which will be true provided that $A < 1$. One can think of the analysis in this section as applying to a two-state system in which one state corresponds to occupancy of the well and the second to escape.

The kinetics of relaxation to equilibrium within the well occurs on a time scale t_r defined in terms of the potential as

$$t_r = \left(\left| \frac{\partial^2 U}{\partial x^2} \right|_{x=1, A=0} \right)^{-1} = 1 \tag{13}$$

Thus, the adiabatic regime is defined by the requirement that $\omega \ll 2\pi/t_r = 2\pi$. In the adiabatic regime the Fokker–Planck equation (3) is characterized by two different time scales, a fast one during which the system relaxes to an "equilibrium," and a slow one determined by the sinusoidal term on the right-hand side of Eq. (3). When the adiabatic approximation is valid one can, at sufficiently short times, use the Kramers approximation to the rate given in Eq. (4) with an effective ΔU containing a time-dependent contribution. This function will be denoted by ΔU_{ad}, which can be written in terms of the stable and unstable points $x_s(t)$ and $x_u(t)$, respectively, as

$$\Delta U_{\text{ad}} = U_{\text{ad}}(x = x_u(t)) - U_{\text{ad}}(x = x_s(t)) \tag{14}$$

The positions of the stable and unstable points are readily determined to be

$$x_u(t) = \frac{1 - [1 + A \cos(\omega t)]^{1/2}}{2}, \qquad x_s(t) = \frac{1 + [1 + A \cos(\omega t)]^{1/2}}{2} \tag{15}$$

After substituting these values into Eq. (4) and performing a small amount of algebra we find as the adiabatic approximation to the escape rate

$$k_{\text{ad}}(\omega t) = \frac{\sqrt{2}}{\pi} [1 + A \cos(\omega t)] \exp\left\{ -\frac{1}{6D} [1 + A \cos(\omega t)]^{3/2} \right\} \tag{16}$$

The probability that the particle is in the well at time t, $S(t)$, can be written approximately as the exponential

$$S(t) \sim \exp\left[-\frac{1}{\omega} \int_0^{\omega t} k_{\text{ad}}(\xi)\, d\xi \right] \tag{17}$$

Provided that one makes the assumption that escape from the potential well is permanent, which is to say that the scale of the escape time is measured by the time to reach the top of the barrier, the function $S(t)$ can be regarded as a survival probability. This, in turn, allows one to define both a probability density for the survival time, $s(t) = -dS/dt$, and an associated survival time $\langle t \rangle$ defined in terms of $S(t)$ as

$$\langle t \rangle = \int_0^\infty S(\tau)\, d\tau \tag{18}$$

We will be interested in the properties of $\langle t \rangle$ as a function of the parameters A, D, and ω.

The numerical calculation of $\langle t \rangle$ can be simplified by reducing it to an integral over a finite interval by taking advantage of the periodicity of $k_{ad}(\omega t)$, i.e., $k_{ad}(\omega t + 2\pi n) = k_{ad}(\omega t)$ for integer n. This allows us to write

$$S\left(t + \frac{2\pi n}{\omega}\right) = S(t) \exp\left(-\frac{2\pi}{\omega} n \langle k \rangle\right) \tag{19}$$

where $\langle k \rangle$ is the average of $k_{ad}(\xi\omega)$ over a single cycle

$$\langle k \rangle = \frac{1}{2\pi} \int_0^{2\pi} k_{ad}(\xi)\, d\xi \tag{20}$$

The relation in Eq. (19) immediately allows us to express $S(t)$ in the form

$$S(t) = e^{-\langle k \rangle t}\, \sigma(t) \tag{21}$$

where $\sigma(t)$ is a nonnegative periodic function of t, the period being $2\pi/\omega$. An exact representation of this function can be found by evaluating the exponent in Eq. (17). A typical curve of the modulating function $\sigma(t)$ as a function of t is shown in Fig. 4. A similar argument justifies writing the probability density for the escape time $s(t) = -dS/dt$ in the same form as Eq. (21), that is, as $\exp(-\langle k \rangle t)\, \eta(t)$ with a second periodic function $\eta(t)$ replacing the function $\sigma(t)$ that occurs in Eq. (21). This new function is not independent of $\sigma(t)$, but can be written in terms of it as

$$\eta(t) = \langle k \rangle\, \sigma(t) - \dot{\sigma}(t) \tag{22}$$

which is also periodic. There is an implicit constraint on the function $\sigma(t)$, since $s(t)$ must be nonnegative. We conclude from this consideration that within the framework of the adiabatic approximation the probability density $s(t)$ can be expected to vary nonmonotonically as a function of t, replacing the monotonic dependence on time that is found in the absence

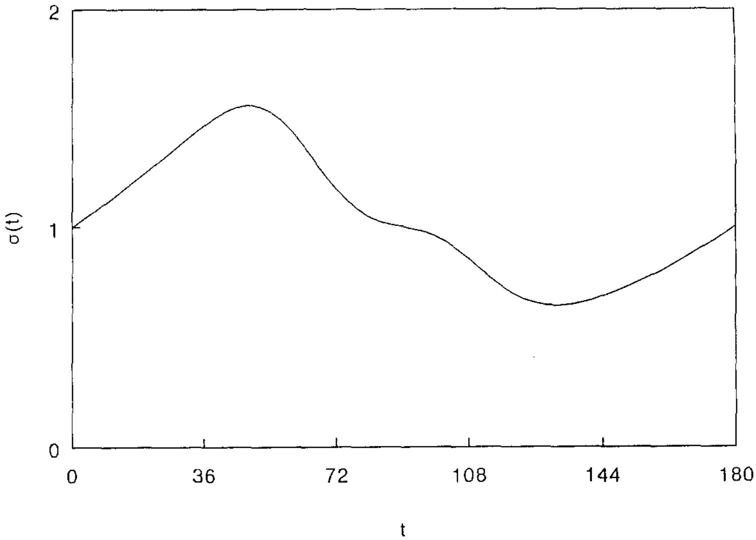

Fig. 4. An illustration of the modulating function $\sigma(t)$ plotted as a function of t.

of an imposed periodic driving force. Another candidate for resonant behavior is the mean survival time $\langle t \rangle$, which can be expressed as

$$\langle t \rangle = \left[1 - \exp\left(-\frac{2\pi}{\omega}\langle k \rangle\right)\right]^{-1} \int_0^{2\pi/\omega} \exp\left[-\frac{1}{\omega}\int_0^{\omega t} k_{ad}(\xi)\, d\xi\right] dt \qquad (23)$$

While it is possible to evaluate the function $k_{ad}(\xi)$ in terms of an infinite series, the results of such a calculation can only be expressed in a very cumbersome and noninformative form. We therefore summarize results that have been obtained numerically, after which we discuss some limiting cases which can be dealt with analytically. Three parameters occur in all of the preceding formulas which may be set arbitrarily: the amplitude of the periodic forcing term $A/4$, the constant D, which measures the strength of the noise, and the frequency ω. Our numerical results indicate that all of the parameters that have been calculated are monotonic functions of D when A is small. However, these parameters can depend nonmonotonically on A. In addition the calculations indicate a very weak dependence on the frequency, hence we will keep ω fixed.

The monotonic dependence of $\langle k \rangle$ on D follows trivially from the combination of Eqs. (16) and (20), since D appears only in the exponent of our expression for $k_{ad}(\xi)$, and this function itself is a monotonic function of D. However, the parameter A appears both in the exponential and the prefactor in Eq. (16), suggesting the possibility that $\langle k \rangle$ might be a nonmonotonic function of the amplitude. Figure 5 shows curves of $\langle k \rangle$ as a function of A. In a sufficiently noisy situation there is no resonance behavior by this function since escape is dominated by the noise. Only at sufficiently small values of D does the resonant effect become evident. If A is very small, it cannot help to push the particle out of the potential well, and all one can hope to observe is the average rate due to noise alone. This must be small since we have assumed that the amplitude of the noise is small. On the other hand, when A is very large the initial swing, now primarily a result of the periodic forcing term, pushes the particle far along the positive branch of the potential. This has the effect of decreasing the average escape rate, since the particle must traverse the entire potential well before reaching the barrier maximum. A straightforward calculation of $\langle k \rangle$ using Laplace's method for the asymptotic evaluation of integrals shows that the value of A that maximizes the average escape rate is

$$A \sim 1 - (6D)^{2/3} \tag{24}$$

and when this value is substituted into the definition of $\langle k \rangle$ in Eq. (20) the average escape rate is found to be proportional to $\exp[-(6D)^{-1/3}]$. The type of resonant effect just described for the parameter $\langle k \rangle$ also appears in $\langle t \rangle$, which exhibits a minimum when considered as a function of A, but is a monotonic function of the noise amplitude D.

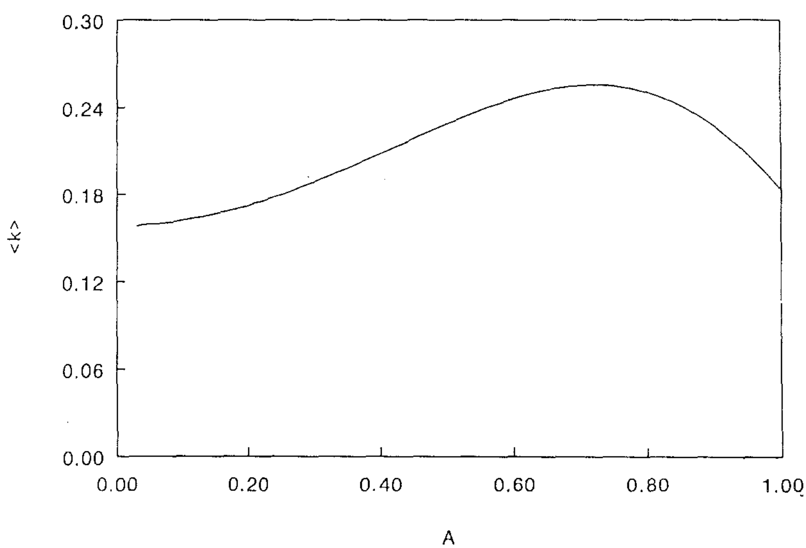

Fig. 5. A curve of the averaged rate constant $\langle k \rangle$ plotted as a function of the amplitude A of the field.

4. THE LIMIT CASES $A/D \gg 1$ AND $A/D \ll 1$

In the last section we introduced the adiabatic approximation to deal with a case in which both the noise amplitude D and the amplitude of the forcing term A are small in comparison to the barrier height. However, the ratio A/D can take on values either very small or very large in comparison with 1. In this section we consider both of these cases.

When $A/D \ll 1$ we may expand $k_{ad}(\xi)$ as

$$k_{ad}(\xi) \sim \frac{\sqrt{2}}{\pi} e^{-1/(6D)} \left(1 - \frac{A}{4D} \cos \xi\right) = k_0 \left(1 - \frac{A}{4D} \cos \xi\right) \tag{25}$$

so that $\langle k \rangle$ is equal to k_0, which is the Kramers form of the rate constant given by Eq. (4), and the survival probability $S(t)$ is found to be approximated by

$$S(t) \sim \exp\left[-k_0 t + \frac{k_0 A}{6D\omega} \sin(\omega t)\right] \tag{26}$$

which indicates a small periodic modulation of an exponential decay. To the same degree of approximation the average $\langle t \rangle$ can be expressed as

$$\langle t \rangle \sim \frac{1}{k_0} \left(1 + \frac{A}{6D} \frac{k_0^2}{k_0^2 + \omega^2}\right) \tag{27}$$

which indicates that the effect of the modulation is to increase the mean escape time. It is also evident from this expression that other than the periodic modulation already noted, there is no resonant effect in this order of approximation, either with respect to the noise amplitude D or as a function of the frequency of the driving field.

Let us next consider the case in which $A < 1$ but $A/D \gg 1$. These conditions allow us to retain the adiabatic approximation with the result that the representation of $S(t)$ in Eq. (17) remains a useful one. In this case it is no longer possible to expand the sinusoidal term in the exponent to evaluate the integral as has been done in the last paragraph. Instead we shall resort to a second strategy for evaluating the integral. The survival probability in the adiabatic approximation together with the assumption that $A/D \gg 1$ yields

$$S(t) \sim \exp\left[-\frac{\sqrt{2}}{\pi\omega} \int_0^{\omega t} \exp(-\lambda \cos \xi)\, d\xi\right] \tag{28}$$

where $\lambda = A/(4D)$. While the integral appearing in this expression can be evaluated in closed form, the resulting expression takes the form of an awkward infinite series whose properties are not readily determined. However, we can take advantage of the fact that $\lambda \gg 1$ to calculate a relatively simple approximation to $S(t)$. It is clear that the value of the integral appearing in the exponent will be dominated by contributions from regions in which $\cos \xi \sim -1$. These occur at $\xi_n = (2n+1)\pi$, where $n = 0, 1, 2, \ldots$. At these points $S(t)$ decreases as a negative exponential of the function $\exp(\lambda)/\sqrt{\lambda}$. Hence we conclude that $S(t)$ is essentially equal to 0 when $t > \pi/\omega$.

Within the interval $(0, \pi/\omega)$ we may derive an approximation to $S(t)$ by noting that the integral will be dominated by the behavior of the integrand in the neighborhood of the point at which $\cos \xi$ is a minimum, i.e., for $\xi \sim \omega t$. If we expand the integrand around this point retaining only the lowest-order term, we find that the interpolating form of $S(t)$ is

$$S(t) \sim \exp\left(-\frac{\sqrt{2}}{\pi \lambda \omega} \{1 - \exp[-\lambda \omega t \sin(\omega t)]\} \right) \tag{29}$$

which will be an accurate approximation except in the immediate neighborhood of the point at which the sharp decrease occurs, i.e., at $\omega t = \pi$. Hence no oscillatory features can be expected when $\lambda \gg 1$.

5. THE HIGH-FREQUENCY LIMIT

The adiabatic approximation is valid provided that ω is small. In the opposite regime ($\omega \gg 1$) we do not expect to see any oscillatory behavior because the highly oscillatory driving force tends to average itself out over the time scale of the secular motion, which is equivalent to the statement that the effect of the oscillatory term should be small in comparison to the dynamical behavior found for the oscillation-free system. A precise specification of the large-ω regime is as that in which $\omega \gg 2\pi/t_r = 2\pi$, where, as before, the value of t_r defines the time scale of oscillations in the potential well. Notice that in the presence of noise there will be three time scales, t_r, ω^{-1}, and the time scale associated with the noise, which will be considered to be negligibly small in comparison to ω^{-1}. Let us neglect the noise for the moment, a step which will not be crucial to our argument. The function $x(t)$ then satisfies the equation

$$\dot{x} = x - x^2 + \frac{A}{4} \cos(\omega t) \tag{30}$$

Let us next decompose $x(t)$ into a sum of two terms as follows:

$$x(t) = X(t) + A \frac{\sin(\omega t)}{4\omega} \tag{31}$$

The first term on the right-hand side will be assumed to vary significantly only over times of the order of t_r, while the second term on the right-hand side is taken to be small in comparison to $X(t)$ because of the factor ω appearing in the denominator. On substituting Eq. (31) into Eq. (30) we find that the equation satisfied by $X(t)$ is

$$\dot{X}=X+A\frac{\sin(\omega t)}{4\omega}-X^2-AX\frac{\sin(\omega t)}{2\omega}-A^2\frac{\sin^2(\omega t)}{16\omega^2} \tag{32}$$

Since X is assumed to vary slowly over times of the order of ω^{-1}, we can replace this function by its average over a single cycle time of the sinusoid. Let us an average be denoted by $\{X\}$ and perform an average over both sides of the preceding equation. Because of the separation of time scales we have, to a good approximation

$$\{X^2\}\sim\{X\}^2 \qquad \text{and} \qquad \{X\sin(\omega t)\}\sim\{X\}\{\sin(\omega t)\}=0 \tag{33}$$

with the result that $\{X\}$ is the solution to the approximate equation

$$\frac{d\{X\}}{dt}=\{X\}-\{X\}^2-\frac{A^2}{32\omega^2} \tag{34}$$

This is equivalent to statement that the potential is to be replaced by an effective potential $U_{\text{eff}}(x)$ which can be written in terms of $U(x)$ as

$$U_{\text{eff}}(x)=U(x)-\frac{A^2}{32\omega^2}x \tag{35}$$

which is to say, the correction to original potential is quite small and has the effect of lowering the energy barrier by a correspondingly small amount. Again we have recourse to the Kramers formalism, finding thereby that in the large-ω regime the approximate rate constant has the form

$$k(\omega)\sim\frac{\sqrt{2}}{\pi}\left(1+\frac{A^2}{4\omega^2}\right)\exp\left[-\frac{1}{6D}\left(1+\frac{A^2}{4\omega^2}\right)^{3/2}\right] \tag{36}$$

In the last three sections we have considered the kinetics resulting from different regimes while retaining the assumption that the parameter A is less than 1. This allows one to obtain approximate results. In the present section we consider the case $A>1$ in which no analytic tools can be called upon to furnish results in closed form, so that we are perforce required to rely on simulations. We will see that there is numerical evidence for the existence of stochastic resonance in the system defined by Eqs. (2) and (3) in the sense that the escape time depends nonmonotonically on the parameter D, which measures the strength of the noise.

6. LARGE-AMPLITUDE PERIODIC FIELDS

In what follows we set the amplitude A equal to 1.2, which corresponds to the large-A regime. It should be noted that when $A > 1$ an actual well will not exist at all times, but it will certainly exist at some times. We have not been able to furnish a rigorous proof for the certainty of escape, although we believe that escape occurs with probability 1. As a practical matter in our calculations we defined escape to mean that the particle reached $x = -10$, since, at least to the accuracy obtainable by the simulation, no particle returned to the well after having reached this point. In all of our simulations the initial position was $x_0 = 1$, that is, the particle initially sat at the bottom of the potential well. The mean first passage times to escape were found by averaging over the result of 500 replications.

Some typical results of our simulations are shown in Fig. 6, where they are plotted as a function of the noise amplitude D. It is evident from the graphs shown there that at selected values of the amplitude of the periodic term an increase in D leads to an increase in $\langle t \rangle$. When D is sufficiently large the average survival time is determined entirely by the noise. Hence curves corresponding to different frequencies tend to coalesce as in the set of curves in Fig. 6. However, when the noise is small the escape time is mainly determined by the externally applied field, and in particular, this

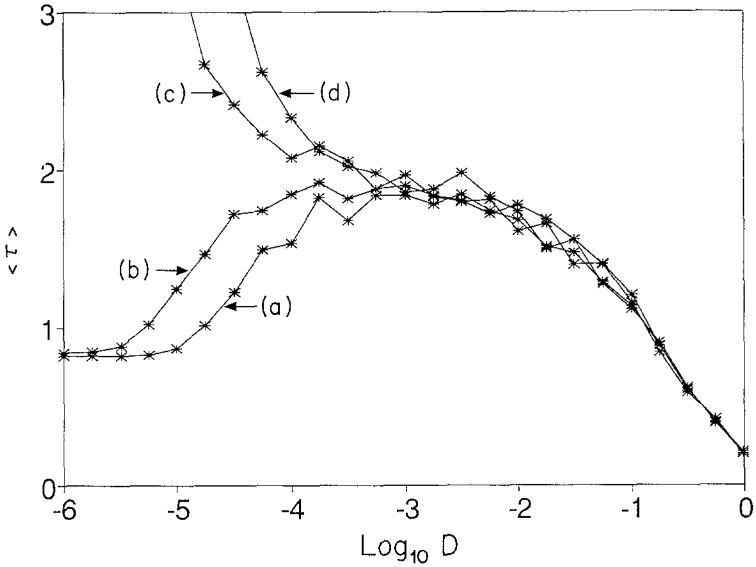

Fig. 6. The average survival time $\langle t \rangle$ plotted as a function of the logarithm of the noise amplitude for $A = 1.2$ and the same frequencies as given in Fig. 2.

time becomes highly sensitive to the imposed frequency. As one can observe from Fig. 6, changes in frequency of the order of 10^{-5} can lead to significant changes in $\langle t \rangle$. This occurs because the frequency shift changes the cycle at which the particle leaves the well.

7. CONCLUSIONS

Resonance in mechanics generally refers to nonmonotonic behavior of a system subjected to a periodic driving force. The most familiar example of this occurs in linear systems when the frequency of the external driving force approaches the natural frequency of the system. Analogous phenomena occur in random systems driven by a periodic field. These effects can be especially pronounced in nonlinear systems.

The term "stochastic resonance" is generally used in a restricted sense to describe nonmonotonic behavior of the Fourier component of the autocorrelation function of the output of the system as a function of the strength of the noise. This phenomenon is expressed as an enhancement of the signal-to-noise ratio, which is a somewhat unintuitive feature of stochastic resonance. There are, however, other manifestations of this phenomenon which have been considered in the literature. For example, we cite the nonmonotomic time-dependent behavior of the probability for dwell times in bistable systems discussed in ref. 5 and analogous effects for the single-well system discussed here. When the amplitude of the periodic field is small the general effect is smeared out by the integration over time in Eq. (23) and the average escape time is a monotonic function of D. This is to be contrasted to the Fourier component of the autocorrelation function, which often remains nonmonotonic even after one integrates over the frequency.

Two different variations of stochastic resonance have been described in this article, based on an examination of parameters which have not been discussed at great length in the literature on this subject. One of these is characterized by the nonmonotonic change with D of what we have termed the average escape time, when the amplitude of the external field is relatively large. Second, we have found a new type of resonance in the escape rate and the average surival time as a function of A. This resonance can exist even when the amplitude A is less than 1 and it depends only peripherally on the fact that the system is also driven by noise.

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